

## Pure spinor superspace identities for massless four-point kinematic factors

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**ABSTRACT:** Using the pure spinor formalism we prove identities which relate the tree-level, one-loop and two-loop kinematic factors for massless four-point amplitudes. From these identities it follows that the complete supersymmetric one- and two-loop amplitudes are immediately known once the tree-level kinematic factor is evaluated. In particular, the two-loop equivalence with the RNS formalism (up to an overall coefficient) is obtained as a corollary.

**KEYWORDS:** Superstrings and Heterotic Strings, Superspaces .

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Massless four-point amplitude at tree-level</b>	<b>2</b>
<b>3. Relating tree-level and one-loop kinematic factors</b>	<b>4</b>
<b>4. Relating one- and two-loop kinematic factors</b>	<b>5</b>
<b>5. The complete tree-level, one- and two-loop kinematic factors</b>	<b>6</b>

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## 1. Introduction

Despite numerous successes in the non-perturbative front, perturbative studies of superstring theory remain an important subject of its own. For example, by computing graviton scattering amplitudes one can catch a glimpse of quantum gravity at work in its deepest realms, from which one can derive superstring modifications of Einstein's equations [1].

Since the birth of the pure spinor formalism [2], the task of computing superstring scattering amplitudes has been made easier [3–5]. Its main advantage over the standard Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS) formulations is due to covariant quantization being possible while having manifest space-time supersymmetry. These properties help simplify the required computations, which can be done efficiently. However, until its origins are fully explained [6–8] and a general proof of equivalence with the RNS and GS formalisms is obtained, it is a good measure to check results against the established ones [9, 10]. As far as scattering amplitudes are concerned, these checks have already successfully been done at tree-level [11], one-loop [12] and two-loops [13].

Although the scattering amplitudes were shown to be equivalent, there is a huge difference in the amount of work required to obtain them, most notably in the two-loop case. The RNS two-loop computations of [10] span hundreds of pages because of complications related to the lack of manifest space-time supersymmetry. The computation of [14] is manifestly supersymmetric and therefore contains the result for all possible combination of external states related by supersymmetry, in deep contrast with the bosonic-only computation of [10].

One of the features of the pure spinor formalism that makes this simplification possible is the *pure spinor superspace* nature of its kinematic factors. In this paper we will explore this property and show how careful manipulations in pure spinor superspace can provide even further simplification of superstring scattering amplitude results. The upshot is that the one- and two-loop kinematic factors are completely determined (for Neveu-Schwarz and Ramond external states) once the tree-level amplitude is evaluated.

The massless four-point kinematic factors for the one- and two-loop amplitudes are given by [3, 12, 13]

$$K_1 = \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle + \text{cycl.}(234), \quad (1.1)$$

$$K_2 = \langle (\lambda \gamma^{mnpqr} \lambda) \mathcal{F}_{mn}^1 \mathcal{F}_{pq}^2 \mathcal{F}_{rs}^3 (\lambda \gamma^s W^4) \rangle \Delta(1, 3) \Delta(2, 4) + \text{perm.}(1234), \quad (1.2)$$

where  $\Delta(i, j) \equiv \Delta(z_i, z_j)$  is the basic biholomorphic 1-form defined in [10]. Interestingly, the analogous expression for the tree-level amplitude has never been found, even after the explicit computations of [15]. So in section 2 we compute the massless four-point amplitude at tree-level to get

$$K_0 = \frac{1}{2} k_1^m k_2^n \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) \mathcal{F}_{mn}^4 \rangle - (k^1 \cdot k^3) \langle A_n^1 (\lambda A^2)(\lambda A^3)(\lambda \gamma^n W^4) \rangle + (1 \leftrightarrow 2). \quad (1.3)$$

Then we proceed to show in sections 3 and 4 that  $K_0$ ,  $K_1$  and  $K_2$  satisfy the following identities

$$K_0 = -\langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle = -\frac{1}{3} K_1 \quad (1.4)$$

$$K_2 = 32 K_0 [(u-t)\Delta(1, 2)\Delta(3, 4) + (s-t)\Delta(1, 3)\Delta(2, 4) + (s-u)\Delta(1, 4)\Delta(2, 3)] \quad (1.5)$$

Finally, in section 5 we explain why some fermionic results reported in [16] contradict the above identities by pointing out a mistake made in [16] which invalidates its conclusions. After clarifying this issue, we compute the tree-level kinematic factor (1.3) for Neveu-Schwarz and Ramond external states, which by the identities (1.4) and (1.5) also completely determine the one- and two-loop kinematic factors simultaneously.

## 2. Massless four-point amplitude at tree-level

Although the massless four-point amplitude at tree-level was already explicitly computed in [15], their derivation overlooked some identities in pure spinor superspace and hid the simplicity of the result. So in this section we compute the closed massless four-point amplitude at tree-level and extract the pure spinor superspace expression for its kinematic factor.

There are at least two motivations to pursue this goal. One is to neatly summarize the whole tree-level computation in one pure spinor superspace expression, which can later be used to find a relation with the one-loop amplitude, as will be done in section 3. The other motivation is related to the ectoplasm method of [17], in which pure spinor superspace expressions in flat space can be used to find supersymmetric invariants in curved space backgrounds.

Following the tree-level prescription of [2, 4], the amplitude to compute is

$$\mathcal{A} = \langle V^1(z_1, \bar{z}_1) V^2(z_2, \bar{z}_2) V^3(z_3, \bar{z}_3) \int_C d^2 z_4 U(z_4, \bar{z}_4) \rangle. \quad (2.1)$$

The closed string vertices are given by the holomorphic square of the open string vertices,  $V(z, \bar{z}) = e^{ik \cdot X} \lambda^\alpha \bar{\lambda}^\beta A_\alpha(\theta) \bar{A}_\beta(\theta)$  and  $U(z, \bar{z}) = e^{ik \cdot X} U(z) \bar{U}(\bar{z})$ , where [3],

$$U(z) = \partial \theta^\alpha A_\alpha(\theta) + A_m(\theta) \Pi^m + d_\alpha W^\alpha(\theta) + \frac{1}{2} N_{mn} \mathcal{F}^{mn}(\theta). \quad (2.2)$$

We note that standard  $SL(2, \mathbb{C})$  invariance allows us to fix  $z_1 = 0, z_2 = 1$  and  $z_3 = \infty$ , so  $\langle \prod_{i=1}^4 : e^{ik^i \cdot X(z_i, \bar{z}_i)} : \rangle = |z_4|^{-\frac{1}{2}\alpha' t} |1 - z_4|^{-\frac{1}{2}\alpha' u} \equiv M(z_4, \bar{z}_4)$ . The first term of (2.2) does not contribute, while the second gives<sup>1</sup>

$$\langle A_m^4 \Pi^m(z_4) \prod_{j=1}^4 : e^{ik^j \cdot X(z_j, \bar{z}_j)} : \rangle = \sum_{j=1}^3 \frac{\alpha'}{2} \frac{ik_j^m}{z_j - z_4} \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A_m^4 \rangle M(z_4, \bar{z}_4). \quad (2.3)$$

Using the standard OPE's [3]

$$N^{mn}(z_4) \lambda^\alpha(z_j) = \frac{\alpha'}{4} \frac{(\lambda \gamma^{mn})^\alpha}{z_j - z_4}, \quad d_\alpha(z_4) V(z_j) = -\frac{\alpha'}{2} \frac{D_\alpha V}{z_j - z_4}, \quad (2.4)$$

we obtain the following OPE identity:

$$\begin{aligned} & \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) \left( d_\alpha(z_4) W_4^\alpha + \frac{1}{2} N^{mn}(z_4) \mathcal{F}_{mn}^4 \right) \rangle \\ &= \frac{\alpha'}{2(z_1 - z_4)} \langle A_m^1 (\lambda A^2)(\lambda A^3) (\lambda \gamma^m W^4) \rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \end{aligned} \quad (2.5)$$

To show this, one uses (2.4) to get

$$\begin{aligned} & \langle (\lambda A^1)(z_1)(\lambda A^2)(z_2)(\lambda A^3)(z_3) d_\alpha(z_4) W_4^\alpha \rangle = \\ &= \frac{\alpha'}{2(z_1 - z_4)} \langle D_\alpha (\lambda A^1)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \end{aligned}$$

Concentrating for simplicity on the first term, the use of the super-Yang-Mills identity  $D_\alpha(\lambda A) = -(\lambda D)A_\alpha + (\lambda \gamma^m)_\alpha A_m$  allows the numerator to be rewritten as

$$\langle D_\alpha (\lambda A^1)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle = -\langle (\lambda D A_\alpha^1)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle + \langle A_m^1 (\lambda A^2)(\lambda A^3) (\lambda \gamma^m W^4) \rangle. \quad (2.6)$$

As BRST-exact terms decouple, the first term in the right hand side of (2.6) becomes

$$\begin{aligned} & -\frac{\alpha'}{2(z_1 - z_4)} \langle (\lambda D A_\alpha^1)(\lambda A^2)(\lambda A^3) W_4^\alpha \rangle = -\frac{\alpha'}{2(z_1 - z_4)} \langle A_\alpha^1 (\lambda A^2)(\lambda A^3) (\lambda D) W_4^\alpha \rangle \\ &= -\frac{\alpha'}{8(z_1 - z_4)} \langle (\lambda \gamma^{mn} A^1)(\lambda A^2)(\lambda A^3) \mathcal{F}_{mn}^4 \rangle. \end{aligned}$$

However, this term is exactly canceled by the  $(z_1 - z_4)^{-1}$  contribution from the OPE

$$\frac{1}{2} \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) (N^{mn} \mathcal{F}_{mn}^4) \rangle = \frac{\alpha'}{8(z_1 - z_4)} \langle (\lambda \gamma^{mn} A^1)(\lambda A^2)(\lambda A^3) \mathcal{F}_{mn}^4 \rangle + \dots,$$

which finishes the proof of (2.5).

With the results (2.3) and (2.5), the correlation in the amplitude (2.1) reduces to

$$\mathcal{A} = \left( \frac{\alpha'}{2} \right)^2 \int_C d^2 z_4 \left( \frac{F_{12}}{z_4} + \frac{F_{21}}{1 - z_4} \right) \left( \frac{\bar{F}_{12}}{\bar{z}_4} + \frac{\bar{F}_{21}}{1 - \bar{z}_4} \right) |z_4|^{-\frac{1}{2}\alpha' t} |1 - z_4|^{-\frac{1}{2}\alpha' u},$$

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<sup>1</sup>To avoid the pollution of notation, we mostly omit the  $z_i$  dependence in the superfields even when computing OPE's, as it does not prevent the proper understanding of the formulæ.

where  $F_{12} = ik_1^m \langle (\lambda A^1)(\lambda A^2)(\lambda A^3)A_m^4 \rangle + \langle A_m^1(\lambda A^2)(\lambda A^3)(\lambda \gamma^m W^4) \rangle$  and  $F_{21}$  is obtained by exchanging  $1 \leftrightarrow 2$ . The integral can be evaluated using the following formula [18]

$$\int_C d^2 z z^N (1-z)^M \bar{z}^{\bar{N}} (1-\bar{z})^{\bar{M}} = 2\pi \frac{\Gamma(1+N)\Gamma(1+M)}{\Gamma(2+N+M)} \frac{\Gamma(-1-\bar{N}-\bar{M})}{\Gamma(-\bar{N})\Gamma(-\bar{M})}.$$

After a few manipulations one finally gets

$$\mathcal{A} = -2\pi \left(\frac{\alpha'}{2}\right)^4 K_0 \bar{K}_0 \frac{\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})\Gamma(-\frac{\alpha' s}{4})}{\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})\Gamma(1+\frac{\alpha' s}{4})},$$

where  $K_0 = \frac{1}{2}(uF_{12} + tF_{21})$  is given by

$$K_0 = \langle (\partial_m A_n^1)(\lambda A^2)\partial^m(\lambda A^3)(\lambda \gamma^n W^4) \rangle - \frac{1}{2} \langle \partial^m(\lambda A^1)\partial^n(\lambda A^2)(\lambda A^3)\mathcal{F}_{mn}^4 \rangle + (1 \leftrightarrow 2), \quad (2.7)$$

which is the sought-for kinematic factor in pure spinor superspace. As will become clear later, it is convenient to rewrite (2.7) without explicit labels,

$$K_0 = 2\langle (\partial_m A_n)(\lambda A)\partial^m(\lambda A)(\lambda \gamma^n W) \rangle - \langle (\lambda A)\partial^m(\lambda A)\partial^n(\lambda A)\mathcal{F}_{mn} \rangle. \quad (2.8)$$

Furthermore, using the identities of [19] we will compute in section 5 the whole component expansion of (2.7) (for Neveu-Schwarz and Ramond external states). We will see that expression (2.7) neatly summarizes the rather lengthy computations of [15].

### 3. Relating tree-level and one-loop kinematic factors

Using the well-known superfield equations of motion in the formulation of ten-dimensional Super-Yang-Mills theory in superspace [20, 21], one can show that

$$Q\mathcal{F}_{mn} = 2\partial_{[m}(\lambda\gamma_{n]}W), \quad QW^\alpha = \frac{1}{4}(\lambda\gamma^{mn})^\alpha \mathcal{F}_{mn}, \quad QA_m = (\lambda\gamma_m W) + \partial_m(\lambda A), \quad (3.1)$$

where  $Q = \oint \lambda^\alpha d_\alpha$  is the pure spinor BRST operator.<sup>2</sup> With these relations in hand we will show that (1.4) holds true. To prove this we note that  $\langle (\lambda A)\partial^m(\lambda A)(QA^n)F_{mn} \rangle = -\langle (\lambda A)\partial^m(\lambda A)A^n(QF_{mn}) \rangle$ , which upon use of (3.1) and momentum conservation becomes

$$\begin{aligned} \langle (\lambda A)\partial^m(\lambda A)(QA^n)F_{mn} \rangle &= \langle (\lambda A)\partial^m(\lambda A)\partial_m A_n(\lambda\gamma^n W) \rangle \\ &\quad - \langle \partial_n(\lambda A)\partial_m(\lambda A)A^n(\lambda\gamma^m W) \rangle - \langle (\lambda A)\partial_n\partial_m(\lambda A)A^n(\lambda\gamma^m W) \rangle. \end{aligned} \quad (3.2)$$

The second term can be rewritten like

$$\langle \partial_n(\lambda A)\partial_m(\lambda A)A^n(\lambda\gamma^m W) \rangle = -\langle (\lambda A)(\lambda\gamma^m W) [A^n\partial_m\partial_n(\lambda A) + \partial^n(\lambda A)\partial_m A_n] \rangle$$

as can be shown by integrating  $\partial^m$  by parts and using the equation of motion for  $W^\alpha$ . So,

$$\langle (\lambda A)\partial^m(\lambda A)(QA^n)F_{mn} \rangle = \langle (\lambda A)\partial^m(\lambda A)(\lambda\gamma^n W)F_{mn} \rangle - 2\langle (\lambda A)\partial_n\partial_m(\lambda A)A^n(\lambda\gamma^m W) \rangle$$

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<sup>2</sup>We refer the reader to the lectures notes in the pure spinor formalism [21] for the basic definitions.

which implies that  $\langle (\lambda A) \partial^m (\lambda A) \partial^n (\lambda A) F_{mn} \rangle = -2 \langle (\lambda A) \partial_n \partial_m (\lambda A) A^n (\lambda \gamma^m W) \rangle$ , or equivalently,

$$\langle (\lambda A) \partial^m (\lambda A) \partial^n (\lambda A) F_{mn} \rangle = -2 \langle (\lambda A) \partial_n (Q A_m) A^n (\lambda \gamma^m W) \rangle. \quad (3.3)$$

Using  $[Q, \partial^n] = 0$  and the decoupling of BRST-trivial operators, equation (3.3) becomes

$$\begin{aligned} \langle (\lambda A) \partial^m (\lambda A) \partial^n (\lambda A) F_{mn} \rangle &= 2 \langle (\lambda A) (\partial_n A_m) (Q A^n) (\lambda \gamma^m W) \rangle \\ &= \langle (\lambda A) (\lambda \gamma^m W) (\lambda \gamma^n W) F_{mn} \rangle + 2 \langle (\partial_n A_m) (\lambda A) \partial^n (\lambda A) (\lambda \gamma^m W) \rangle. \end{aligned} \quad (3.4)$$

Plugging (3.4) in the tree-level kinematic factor (2.8) we finally obtain

$$K_0 = - \langle (\lambda A) (\lambda \gamma^m W) (\lambda \gamma^n W) F_{mn} \rangle = -\frac{1}{3} K_1, \quad (3.5)$$

which finishes<sup>3</sup> the proof of (1.4).

#### 4. Relating one- and two-loop kinematic factors

To obtain a relation between the one- and two-loop kinematic factors we first need to show that  $\langle (\lambda A^1) (\lambda \gamma^m W^2) (\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle$  is completely symmetric in the labels (1234). This can be done by noting that,<sup>4</sup>

$$\langle (\lambda \gamma^{mnpqr} \lambda) (\lambda A^1) (W^2 \gamma_{pqr} W^3) \mathcal{F}_{mn}^4 \rangle = 4 \langle (\lambda A^1) Q [(W^2 \gamma_{pqr} W^3)] (\lambda \gamma^{pqr} W^4) \rangle. \quad (4.1)$$

Together with the identities  $(\lambda \gamma^{mn} \gamma^{pqr} W^2) (\lambda \gamma_{pqr} W^4) = -48 (\lambda \gamma^{[m} W^2) (\lambda \gamma^{n]} W^4)$  and  $(\lambda \gamma^{mnpqr} \lambda) (W^2 \gamma_{pqr} W^3) = -96 (\lambda \gamma^{[m} W^2) (\lambda \gamma^{n]} W^3)$ , equation (4.1) implies that

$$\begin{aligned} &\langle (\lambda A^1) (\lambda \gamma^m W^4) (\lambda \gamma^n W^2) \mathcal{F}_{mn}^3 \rangle + \langle (\lambda A^1) (\lambda \gamma^m W^3) (\lambda \gamma^n W^4) \mathcal{F}_{mn}^2 \rangle \\ &= 2 \langle (\lambda A^1) (\lambda \gamma^m W^2) (\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle. \end{aligned} \quad (4.2)$$

From (4.2) it follows that,

$$K_{1\text{-loop}} = 3 \langle (\lambda A^1) (\lambda \gamma^m W^2) (\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle. \quad (4.3)$$

Furthermore, the independence of which vertex operator we choose to be non-integrated [11] implies total symmetry of  $\langle (\lambda A^1) (\lambda \gamma^m W^2) (\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle$  in the labels (1234).

Now we can relate the one- and two-loop kinematic factors by noting that

$$\begin{aligned} (\lambda \gamma^{mnpqr} \lambda) \mathcal{F}_{mn}^1 \mathcal{F}_{pq}^2 \mathcal{F}_{rs}^3 (\lambda \gamma^s W^4) &= -4Q [(\lambda \gamma^r \gamma^{mn} W^2) (\lambda \gamma^s W^4) \mathcal{F}_{mn}^1 \mathcal{F}_{rs}^3] \\ &\quad + 8ik_m^1 (\lambda \gamma_n W^1) (\lambda \gamma^r \gamma^{mn} W^2) (\lambda \gamma^s W^4) \mathcal{F}_{rs}^3, \end{aligned} \quad (4.4)$$

<sup>3</sup>This proof was completed a few days after being told that Paul Howe had independently shown the same thing [22].

<sup>4</sup>I thank Nathan Berkovits for suggesting (4.1) to me.

where the pure spinor constraint  $(\lambda\gamma^m\lambda) = 0$  and the identity  $\eta_{mn}\gamma_{\alpha(\beta}^m\gamma_{\gamma\delta)}^n = 0$  must be used to show the vanishing of terms containing factors of  $(\lambda\gamma^m)_{\alpha}(\lambda\gamma_m)_{\beta}$ . Furthermore, as BRST-exact terms decouple from pure spinor correlations  $\langle \dots \rangle$ , equation (4.4) implies

$$\langle (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}^1\mathcal{F}_{pq}^2\mathcal{F}_{rs}^3(\lambda\gamma^sW^4) \rangle = -16ik_m^1\langle (\lambda\gamma^rW^1)(\lambda\gamma^mW^2)(\lambda\gamma^sW^4)\mathcal{F}_{rs}^3 \rangle, \quad (4.5)$$

where we have used  $k_m^1(\lambda\gamma_nW^1)(\lambda\gamma^r\gamma^{mn}W^2) = -2k_m^1(\lambda\gamma^rW^1)(\lambda\gamma^mW^2)$ , which is valid when the equation of motion  $k_m^1(\gamma^mW^1)_{\alpha} = 0$  is satisfied.

Using  $(\lambda\gamma_mW^2) = QA_m^2 - ik_m^2(\lambda A^2)$  and  $\langle (\lambda\gamma^rW^1)Q(A_2^m)(\lambda\gamma^sW^4)\mathcal{F}_{rs}^3 \rangle = 0$  we arrive at the following pure spinor superspace identity

$$\langle (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}_{mn}^1\mathcal{F}_{pq}^2\mathcal{F}_{rs}^3(\lambda\gamma^sW^4) \rangle = 16(k^1 \cdot k^2)\langle (\lambda A^2)(\lambda\gamma^rW^1)(\lambda\gamma^sW^4)\mathcal{F}_{rs}^3 \rangle \quad (4.6)$$

Multiplying (4.6) by  $\Delta(1,3)\Delta(2,4)$  and summing over permutations leads to the following identity,

$$K_2 = -\frac{32}{3}K_1 [(u-t)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(2,4) + (s-u)\Delta(1,4)\Delta(2,3)], \quad (4.7)$$

where we used (4.3) and the standard Mandelstam variables  $s = -2(k^1 \cdot k^2)$ ,  $t = -2(k^1 \cdot k^4)$ ,  $u = -2(k^2 \cdot k^4)$ .

In view of the results in section 5, (4.7) not only provides a simple proof of two-loop equivalence with the (bosonic) RNS result of [10] but it also automatically implies the knowledge of the full amplitude, including fermionic external states.

## 5. The complete tree-level, one- and two-loop kinematic factors

The fermionic results reported in the first version of [16] are in direct contradiction with the identities (4.3) and (4.7). The two-loop kinematic factor (for 2F2B) was incorrectly argued to not have the simple form of (4.7) and the 2F2B one-loop computation of [16] does not obey identity (4.3). We clarify these issues by pointing out the mistake made in [16] which, strictly speaking, invalidates all its fermionic computations. After these issues are settled in the next paragraph, we compute the whole component expansion of (2.7), as that will automatically imply the full knowledge of  $K_2$  for the first time. This is a remarkable example of the simplifying power of the pure spinor formalism.

In [16] the first component of  $W^{\alpha}(\theta)$ , denoted by  $u^{\alpha}$ , is considered to be bosonic instead of fermionic. So, in a strict sense, all fermionic computations in [16] are unreliable and all discussions based on symmetry properties of fermionic kinematic factors need review. In particular, the discussion of the 2F2B kinematic factor at two-loops is wrong because we have proven in (4.7) that it is in fact proportional to the one-loop result.

This symmetry mistake in [16] is also apparent in its computation of  $K_1^{2F2B}$  at one-loop. One can check it in the first formula of section 3.3, where the factor  $(1 - \pi_{34})$  should be  $(1 + \pi_{34})$ . To see this note that the one-loop kinematic factor, with the cyclic permutations written out explicitly,

$$K_1 = \langle (\lambda A^1)(\lambda\gamma^mW^2)(\lambda\gamma^nW^3)\mathcal{F}_{mn}^4 \rangle + \langle (\lambda A^1)(\lambda\gamma^mW^4)(\lambda\gamma^nW^2)\mathcal{F}_{mn}^3 \rangle + \langle (\lambda A^1)(\lambda\gamma^mW^3)(\lambda\gamma^nW^4)\mathcal{F}_{mn}^2 \rangle, \quad (5.1)$$

can be rewritten as

$$K_1 = \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle + \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^4) \mathcal{F}_{mn}^3 \rangle + \langle (\lambda A^1)(\lambda \gamma^m W^3)(\lambda \gamma^n W^4) \mathcal{F}_{mn}^2 \rangle, \quad (5.2)$$

because  $(\lambda \gamma^m W^4)(\lambda \gamma^n W^2) = -(\lambda \gamma^n W^2)(\lambda \gamma^m W^4)$  due to the fermionic nature of  $W^\alpha(\theta)$ . So the first line of equation (5.2) can be written in terms of the permutation symbol  $\pi_{ij}$  as  $(1 + \pi_{34}) \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle$ , and not with  $(1 - \pi_{34})$  like shown<sup>5</sup> in [16]. Because of this mistake, the conclusion reached in [16] was that the first line of (5.1) vanished instead of being  $+2 \langle (\lambda A^1)(\lambda \gamma^m W^3)(\lambda \gamma^n W^4) \mathcal{F}_{mn}^2 \rangle$ , as one would conclude by using the identities proven in section 4. In fact, we know that all three terms in (5.1) are equal because of the total symmetry property demonstrated in this paper. So if one subtracts two of them the answer must be zero. The fact that in [16] the author concludes that  $(1 - \pi_{34}) \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) \mathcal{F}_{mn}^4 \rangle = 0$  indicates that its computer codes are indeed correct.

In the sequence we use the following  $\mathcal{N} = 1$  super-Yang-Mills  $\theta$  expansions

$$\begin{aligned} A_\alpha(x, \theta) &= \frac{1}{2} a_m (\gamma^m \theta)_\alpha - \frac{1}{3} (\xi \gamma_m \theta) (\gamma^m \theta)_\alpha - \frac{1}{32} F_{mn} (\gamma_p \theta)_\alpha (\theta \gamma^{mnp} \theta) \\ &\quad + \frac{1}{60} (\gamma_m \theta)_\alpha (\theta \gamma^{mnp} \theta) (\partial_n \xi \gamma_p \theta) + \dots \\ A_m(x, \theta) &= a_m - (\xi \gamma_m \theta) - \frac{1}{8} (\theta \gamma_m \gamma^{pq} \theta) F_{pq} + \frac{1}{12} (\theta \gamma_m \gamma^{pq} \theta) (\partial_p \xi \gamma_q \theta) + \dots \\ W^\alpha(x, \theta) &= \xi^\alpha - \frac{1}{4} (\gamma^{mn} \theta)^\alpha F_{mn} + \frac{1}{4} (\gamma^{mn} \theta)^\alpha (\partial_m \xi \gamma_n \theta) + \frac{1}{48} (\gamma^{mn} \theta)^\alpha (\theta \gamma_n \gamma^{pq} \theta) \partial_m F_{pq} + \dots \\ \mathcal{F}_{mn}(x, \theta) &= F_{mn} - 2(\partial_{[m} \xi \gamma_{n]}) + \frac{1}{4} (\theta \gamma_{[m} \gamma^{pq} \theta) \partial_{n]} F_{pq} + \dots, \end{aligned}$$

and the pure spinor superspace identities in the appendix of [19]. Here  $\xi^\alpha(x) = \chi^\alpha e^{ik \cdot x}$  and  $a_m(x) = e_m e^{ik \cdot x}$  describe the gluino and gluon respectively, while  $F_{mn} = 2\partial_{[m} a_{n]}$  is the gluon field-strength.

Now we compute the whole component expansion of

$$K_0 = \frac{1}{2} k_1^m k_2^n \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) \mathcal{F}_{mn}^4 \rangle - (k^1 \cdot k^3) \langle A_m^1(\lambda A^2)(\lambda A^3)(\lambda \gamma^m W^4) \rangle + (1 \leftrightarrow 2). \quad (5.3)$$

The first term doesn't contribute in the computation of  $K_0(f_1 f_2 f_3 f_4) \equiv K_0^{4F}$ , while the second leads to<sup>6</sup>

$$\begin{aligned} K_0^{4F} &= -\frac{1}{9} (k^1 \cdot k^3) \langle (\lambda \gamma^a \theta)(\lambda \gamma^b \theta)(\lambda \gamma^c \chi^4)(\chi^3 \gamma^b \theta)(\theta \gamma^c \chi^1)(\chi^2 \gamma^a \theta) \rangle + (1 \leftrightarrow 2), \\ &= \frac{1}{5760} \left[ (\chi^1 \gamma^m \chi^2)(\chi^3 \gamma_m \chi^4) [(k^2 \cdot k^3) - (k^1 \cdot k^3)] - \frac{1}{12} (k^3 \cdot k^4) (\chi^1 \gamma^{mnp} \chi^2) (\chi^3 \gamma_{mnp} \chi^4) \right]. \end{aligned}$$

<sup>5</sup>We also note that it is not necessary to make distinctions between  $W^{(\text{even})}$  and  $W^{(\text{odd})}$  to obtain (5.2). One can choose which superfields contribute with fermions ( $\chi^\alpha$ ) or bosons ( $e_m$ ) after performing these pure spinor superspace manipulations.

<sup>6</sup>I acknowledge the use of the GAMMA package [23] in these computations.



Using the following Fierz identity

$$(\chi^1 \gamma^{mnp} \chi^2)(\chi^3 \gamma_{mnp} \chi^4) = 24(\chi^1 \gamma^m \chi^3)(\chi^2 \gamma_m \chi^4) - 12(\chi^1 \gamma^m \chi^2)(\chi^3 \gamma_m \chi^4),$$

we arrive at

$$K_0^{4F} = -\frac{1}{2880} [(k^1 \cdot k^3)(\chi^1 \gamma^m \chi^2)(\chi^3 \gamma_m \chi^4) + (k^3 \cdot k^4)(\chi^1 \gamma^m \chi^3)(\chi^2 \gamma_m \chi^4)]. \quad (5.4)$$

Both terms of (5.3) contribute in the  $K_0^{2B2F} \equiv K_0(f_1 f_2 b_3 b_4)$  kinematic factor,

$$\begin{aligned} K_0^{2B2F} &= -\frac{1}{36} k_1^m k_2^n F_{mn}^4 e_p^3 \langle (\lambda \gamma^t \theta)(\lambda \gamma^u \theta)(\lambda \gamma^p \theta)(\theta \gamma_t \chi^1)(\chi^2 \gamma_u \theta) \rangle \\ &\quad - \frac{1}{24} (k^1 \cdot k^3) F_{mn}^4 e_p^3 \langle (\lambda \gamma^t \theta)(\lambda \gamma^p \theta)(\lambda \gamma^q \gamma^{mn} \theta)(\theta \gamma_q \chi^1)(\chi^2 \gamma_t \theta) \rangle + (1 \leftrightarrow 2) \\ &= \frac{1}{5760} F_{mn}^4 e_p^3 \left[ k_1^m k_2^n (\chi^1 \gamma^p \chi^2) + \frac{1}{2} (k^1 \cdot k^3) (\chi^1 \gamma^{mn} \gamma^p \chi^2) \right] + (1 \leftrightarrow 2) \end{aligned} \quad (5.5)$$

It is worth noticing that the explicit computation of  $K_0^{2B2F}$  becomes easier if we use the identity (3.5) with a convenient choice for the labels in the right hand side, namely  $K_0 = -\langle (\lambda A^1)(\lambda \gamma^m W^3)(\lambda \gamma^n W^4) \mathcal{F}_{mn}^2 \rangle$ , because now one can check that only one term contributes

$$\begin{aligned} K_0^{2B2F} &= \frac{1}{24} \langle (\lambda \gamma^p \theta)(\lambda \gamma^{[m} \gamma^{rs} \theta)(\lambda \gamma^{n]} \gamma^{tu} \theta)(\theta \gamma_p \chi^1)(\chi^2 \gamma_n \theta) \rangle k_m^2 F_{rs}^3 F_{tu}^4 \\ &= \frac{1}{5760} F_{mn}^3 F_{rs}^4 \left[ -i(\chi^1 \gamma^r \chi^2) \eta^{sm} k_2^n + \frac{i}{2} (\chi^1 \gamma^{mnr} \chi^2) k_2^s \right] + (3 \leftrightarrow 4). \end{aligned} \quad (5.6)$$

One can verify that (5.5) and (5.6) are in fact equal and equivalent to the RNS result (see for example [24]). This equality can also be regarded as a check of identity (3.5), which is reassuring. The computation of  $K_0^{4B}$  is straightforward (and can also be deduced from the one-loop result of [12]). One can in fact check that

$$\begin{aligned} K_0^{4B} &= \frac{1}{5760} \left[ -\frac{1}{2} (e^1 \cdot e^3)(e^2 \cdot e^4) ts - \frac{1}{2} (e^1 \cdot e^4)(e^2 \cdot e^3) us - \frac{1}{2} (e^1 \cdot e^2)(e^3 \cdot e^4) tu \right. \\ &\quad + (k^4 \cdot e^1)(k^2 \cdot e^3)(e^2 \cdot e^4) s + (k^3 \cdot e^2)(k^1 \cdot e^4)(e^1 \cdot e^3) s \\ &\quad + (k^3 \cdot e^1)(k^2 \cdot e^4)(e^2 \cdot e^3) s + (k^4 \cdot e^2)(k^1 \cdot e^3)(e^1 \cdot e^4) s \\ &\quad + (k^1 \cdot e^2)(k^3 \cdot e^4)(e^1 \cdot e^3) t + (k^4 \cdot e^3)(k^2 \cdot e^1)(e^2 \cdot e^4) t \\ &\quad + (k^4 \cdot e^2)(k^3 \cdot e^1)(e^3 \cdot e^4) t + (k^1 \cdot e^3)(k^2 \cdot e^4)(e^1 \cdot e^2) t \\ &\quad + (k^2 \cdot e^1)(k^3 \cdot e^4)(e^2 \cdot e^3) u + (k^4 \cdot e^3)(k^1 \cdot e^2)(e^1 \cdot e^4) u \\ &\quad \left. + (k^4 \cdot e^1)(k^3 \cdot e^2)(e^3 \cdot e^4) u + (k^2 \cdot e^3)(k^1 \cdot e^4)(e^1 \cdot e^2) u \right] \\ &= \frac{1}{2880} t_8^{m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} F_{m_1 n_1}^1 F_{m_2 n_2}^2 F_{m_3 n_3}^3 F_{m_4 n_4}^4, \end{aligned} \quad (5.7)$$

where we used the  $t_8$  tensor definition of [25, 26].

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